Stringy E-functions of varieties with A-D-E singularities

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Abstract

The stringy E-function for normal irreducible complex varieties with at worst log terminal singularities was introduced by Batyrev. It is defined by data from a log resolution. If the variety is projective and Gorenstein and the stringy E-function is a polynomial, Batyrev also defined the stringy Hodge numbers as a generalization of the Hodge numbers of nonsingular projective varieties, and conjectured that they are nonnegative. We compute explicit formulae for the contribution of an A-D-E singularity to the stringy E-function in arbitrary dimension. With these results we can say when the stringy E-function of a variety with such singularities is a polynomial and in that case we prove that the stringy Hodge numbers are nonnegative.

1 Introduction

1.1. In [Ba1], Batyrev defined the stringy E-function for normal irreducible complex algebraic varieties, with at worst log terminal singularities. With this function he was able to formulate a topological mirror symmetry test for Calabi-Yau varieties with singularities. Before stating the definition of the stringy E-function, we recall some other definitions.

Let X be a complex algebraic variety. One defines the Hodge-Deligne polynomial $H(X;u,v) \in \mathbb{Z}[u,v]$ by

$$H(X;u,v) = \sum_{i=0}^{2d} (-1)^i \sum_{p,q} h^{p,q}(H_c^i(X,\mathbb{C}))u^p v^q,$$

where $h^{p,q}$ denotes the dimension of the (p,q)-component of the mixed Hodge structure on $H_c^i(X,\mathbb{C})$. A nice introduction to Deligne's mixed Hodge theory and to this definition can be found in [Sr] (pay attention to the extra factor

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$(-1)^{p+q}$ that the author has inserted there). The Hodge-Deligne polynomial is a generalized Euler characteristic, that is, it satisfies:

(1) $H(X) = H(X \setminus Y) + H(Y)$ where Y is Zariski-closed in X,

(2) $H(X \times X') = H(X) \cdot H(X')$.

Note that $H(X; 1, 1) = \chi(X)$, the topological Euler characteristic of X.

1.2. A normal irreducible complex variety X is called $\mathbb{Q}$-Gorenstein if rK_X is Cartier for some $r \in \mathbb{Z}_{>0}$. Take a log resolution $\varphi : \widetilde{X} \to X$ (i.e. a proper birational morphism from a nonsingular variety $\widetilde{X}$ such that the exceptional locus of φ is a divisor whose components $D_1, \dots, D_s$ are smooth and have normal crossings). Then we have $rK_{\widetilde{X}} - \varphi^*(rK_X) = \sum_i b_i D_i$, with $b_i \in \mathbb{Z}$. This is also formally written as $K_{\widetilde{X}} - \varphi^*(K_X) = \sum_i a_i D_i$, where $a_i = \frac{b_i}{r}$. The variety X is called terminal, canonical, log terminal and log canonical if $a_i > 0, a_i \geq 0, a_i > -1, a_i \geq -1$, respectively, for all i (this is independent of the chosen log resolution). The difference $K_{\widetilde{X}} - \varphi^*(K_X)$ is called the *discrepancy*.

1.3. Now we are ready to define the stringy E-function. We discuss its properties and give the additional definitions of the stringy Euler number and the stringy Hodge numbers. All of this goes back to Batyrev [Ba1].

Definition. Let X be a normal irreducible complex variety with at most log terminal singularities and let $\varphi : \widetilde{X} \to X$ be a log resolution. Denote the irreducible components of the exceptional locus by D_i, $i \in I$, and write D_J for $\cap_{j \in J} D_j$ and D_J° for $D_J \setminus \cup_{j \in I \setminus J} D_j$, where J is any subset of I ($D_\emptyset$ is taken to be $\widetilde{X}$). The stringy E-function of X is

$$E_{st}(X; u, v) := \sum_{J \subseteq I} H(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j + 1} - 1},$$

where a_j is the discrepancy coefficient of D_j and where the product $\prod_{j \in J}$ is 1 if $J = \emptyset$.

Batyrev proved that this definition is independent of the chosen log resolution. His proof uses motivic integration. An overview of this theory is provided in [Ve1].

Remark.

(1) If X is smooth, then $E_{st}(X) = H(X)$ and if X admits a crepant resolution $\varphi : \widetilde{X} \to X$ (i.e. such that the discrepancy is 0), then $E_{st}(X) = H(\widetilde{X})$.

(2) If X is Gorenstein (i.e. K_X is Cartier), then all $a_i \in \mathbb{Z}_{\geq 0}$ and $E_{st}(X)$ becomes a rational function in u and v. It is then an element of $\mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v)$.

(3) The *stringy Euler number* of X is defined as

$$\lim_{u,v\to 1} E_{st}(X;u,v) = \sum_{J\subseteq I} \chi(D_J^\circ) \prod_{j\in J} \frac{1}{a_j+1}.$$

1.4. Assume moreover that X is projective of dimension d. Then Batyrev proved the following instance of Poincaré and Serre duality:

(i) $E_{st}(X;u,v) = (uv)^d E_{st}(X;u^{-1},v^{-1})$,

(ii) $E_{st}(X;0,0) = 1$.

If X has at worst Gorenstein canonical singularities and if $E_{st}(X;u,v)$ is a polynomial $\sum_{p,q} a_{p,q}u^p v^q$, he defined the *stringy Hodge numbers* of X as $h_{st}^{p,q}(X) := (-1)^{p+q}a_{p,q}$. It is clear that

(1) they can only be nonzero for $0 \le p,q \le d$,

(2) $h_{st}^{0,0} = h_{st}^{d,d} = 1$,

(3) $h_{st}^{p,q} = h_{st}^{q,p} = h_{st}^{d-p,d-q} = h_{st}^{d-q,d-p}$,

(4) if X is smooth, the stringy Hodge numbers are equal to the usual Hodge numbers.

Conjecture (Batyrev). *The stringy Hodge numbers are nonnegative.*

Example. The conjecture is true for varieties that admit a crepant resolution. This is the case for all canonical surface singularities, which are exactly the two-dimensional A-D-E singularities [Re, p.375] (see also Theorem 5.1 for $m=3$).

Remark. For a complete surface X with at most log terminal singularities, Veys showed that

$$E_{st}(X) = \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} h_{st}^{p,q} u^p v^q + \sum_{r\notin\mathbb{Z}} h_{st}^{r,r}(uv)^r,$$

with all $h_{st}^{p,q}$ and $h_{st}^{r,r}$ nonnegative [Ve2, p.138].

1.5. In this paper, we will compute in arbitrary dimension the contribution of an A-D-E singularity to the stringy E-function. This has already been done by Dais and Roczen in the three-dimensional case (see [DR]), but their computation of some discrepancy coefficients in the D and E cases is inaccurate and this leads to incorrect formulae in these cases. We correct and considerably simplify their formulae (also for type A). We construct a log resolution for all higher dimensional A-D-E singularities (based on the calculation by Dais and Roczen of a log resolution for the three-dimensional A-D-E's), and again we are always able

to obtain a fairly simple formula for their stringy E-function. For the contribution of an $(m-1)$-dimensional singularity of type D_n (where m is odd and $n = 2k$ is even) we find for example

$$1 + \frac{(uv-1)}{((uv)^{(2k-1)(m-3)+1} - 1)} \left(\sum_{i=1}^{2k-1} (uv)^{i(m-3)+1} + (uv)^{k(m-3)+1} \right).$$

Then using our concrete formulae, we can prove the following theorem.

Theorem. *Let X be a projective complex variety of dimension at least 3 with at most A-D-E singularities. The stringy E-function of X is a polynomial if and only if X has dimension 3 and all singularities are of type A_n (n odd) and/or D_n (n even). In that case, the stringy Hodge numbers of X are positive.*

In the next section we recall the definition of the A-D-E singularities and we construct a log resolution for them. In section 3 and 4, we compute the Hodge-Deligne polynomials and the discrepancy coefficients that we need, respectively. In section 5 we give the resulting formulae and prove the theorem.

2 A-D-E singularities and their desingularization

2.1. Definition. By a d-dimensional ($d \geq 2$) A-D-E singularity we mean a singularity that is analytically isomorphic to the germ at the origin of one of the following hypersurfaces in $\mathbb{A}_{\mathbb{C}}^{d+1}$ (with coordinates $(x_1, \ldots, x_{d+1})$):

(1) $x_1^{n+1} + x_2^2 + x_3^2 + \cdots + x_{d+1}^2 = 0$ (type $A_n, n \geq 1$),

(2) $x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_{d+1}^2 = 0$ (type $D_n, n \geq 4$),

(3) $x_1^3 + x_2^4 + x_3^2 + \cdots + x_{d+1}^2 = 0$ (type E_6),

(4) $x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_{d+1}^2 = 0$ (type E_7),

(5) $x_1^3 + x_2^5 + x_3^2 + \cdots + x_{d+1}^2 = 0$ (type E_8).

Some of their properties are listed in [DR, Remark 1.10].

2.2. We will now construct a log resolution for these singularities by performing successive blow-ups, but we will only do this for $d \geq 4$. The case $d = 2$ is well known and the construction in the three-dimensional case can be found in detail

in [DR, Section 2]; in fact, our procedure is quite analogous. The main differences are:

(1) For $d \geq 4$, every blow-up adds just one component to the exceptional locus, whereas you can get two planes intersecting in a line as new exceptional divisors after a single blow-up in the three-dimensional case (e.g. after the first blow-up in cases D and E).

(2) In the higher dimensional case, the analogue of this line will be a singular line on the exceptional divisor, thus in order to get a smooth normal crossings divisor one has to blow up in such lines, which is not necessary for $d = 3$.

An example will make this clear: blow up in the singular point of the defining hypersurface in the E_6 case. For a suitable choice of coordinates one finds $\{z_3^2 + z_4^2 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ as equation of the exceptional locus for $d = 3$, and for $d \geq 4$, one finds $\{z_3^2 + z_4^2 + \cdots + z_{d+1}^2 = 0\} \subset \mathbb{P}^d_{\mathbb{C}}$ (this is irreducible, but the line $\{z_3 = \cdots = z_{d+1} = 0\}$ is singular).

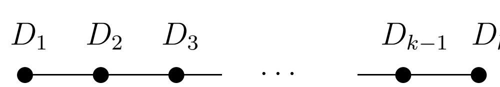
In what follows we use the same name for a divisor D at the moment of its creation as at all later stages (instead of speaking of the strict transform of D). We work out the details for the case of a D_n singularity with even n and we discuss the results shortly in the other cases. We write m for the number of variables ($m \geq 5$) and use coordinates $(x_1, \ldots, x_m)$ on $\mathbb{A}^m$.

2.3. $\boxed{\textbf{Case A}}$

Consider the hypersurface $X = \{x_1^{n+1} + x_2^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$ for $m \geq 5$.

(1) n odd, $n = 2k - 1$, with $k \geq 1$.

Blowing up an A_n singularity yields an A_{n-2} singularity (that lies on the exceptional locus) and nothing else happens. Thus after k point blow-ups we already have a log resolution. The intersection diagram looks like



where D_i is created after the i-th blow-up. At the moment of its creation, D_i (for $i \in \{1, \ldots, k-1\}$) is isomorphic to the singular quadric $\{x_2^2 + \cdots + x_m^2 = 0\}$ in $\mathbb{P}^{m-1}$, and its singular point is the center of the next blow-up. The last divisor D_k is isomorphic to the nonsingular quadric in $\mathbb{P}^{m-1}$. In the end the intersection of two exceptional divisors is isomorphic to a nonsingular quadric in $\mathbb{P}^{m-2}$.

(2) n even, $n = 2k$, with $k \geq 1$.

After k point blow-ups the strict transform of X is nonsingular, but the last created divisor D_k still has a singular point, so we have to perform an extra blow-up (with exceptional divisor D_{k+1} isomorphic to $\mathbb{P}^{m-2}$). As intersection diagram we find

with all D_i ($i \in \{1, \ldots, k\}$) isomorphic to the singular quadric $\{x_2^2 + \cdots + x_m^2 = 0\}$ in $\mathbb{P}^{m-1}$ at the moment of their creation. Again, all intersections are isomorphic to the nonsingular quadric in $\mathbb{P}^{m-2}$.

2.4. $\boxed{\textbf{Case D}}$

Now we study $X = \{x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$ for $m \geq 5$ and $n \geq 4$. Notice that you also find singularities for $n = 2$ and $n = 3$, but they are analytically isomorphic to two A_1 and one A_3 singularity respectively.

(1) n even, $n = 2k$, with $k \geq 2$.

Step 1: We blow up X in the origin. Take $(x_1, \ldots, x_m) \times (z_1, \ldots, z_m)$ as coordinates on $\mathbb{A}^m \times \mathbb{P}^{m-1}$. Consider the reducible variety X' in $\mathbb{A}^m \times \mathbb{P}^{m-1}$ given by the equations

$$\begin{cases} x_1^{2k-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0 & \\ x_i z_j = x_j z_i & \forall\, i, j \in \{1, \ldots, m\}. \end{cases}$$

In the open set $z_1 \neq 0$, X' is isomorphic to $\{x_1^2(x_1^{2k-3} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2) = 0\} \subset \mathbb{A}^m$ by replacing x_j by $x_1 \frac{z_j}{z_1}$ and renaming the affine coordinate $\frac{z_j}{z_1}$ as x_j for $j = 2, \ldots, m$. The equation $x_1 = 0$ describes here the exceptional locus, while the other equation gives us the strict transform of X, in which we are interested. Their intersection is the first exceptional divisor, we call it D_1. We can do the same thing for any open set $z_i \neq 0$ and thus we can describe X' by the following set of equations:

$$\begin{cases} x_1^2(x_1^{2k-3} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2) = 0 & (1) \\ x_2^2(x_1^{2k-1} x_2^{2k-3} + x_1 x_2 + x_3^2 + \cdots + x_m^2) = 0 & (2) \\ x_3^2(x_1^{2k-1} x_3^{2k-3} + x_1 x_2^2 x_3 + 1 + x_4^2 + \cdots + x_m^2) = 0 & (3) \\ \quad\vdots & \vdots \\ x_m^2(x_1^{2k-1} x_m^{2k-3} + x_1 x_2^2 x_m + x_3^2 + \cdots + x_{m-1}^2 + 1) = 0. & (m) \end{cases}$$

One sees from this that globally $D_1 \cong \{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, which has a singular line $\{x_3 = \cdots = x_m = 0\}$ (located in charts (1) and (2)). Notice

that for $k \geq 3$, we have a D_{n-2} singularity in chart (1) and a singularity that is analytically isomorphic to an A_1 in the origin of chart (2). In the other charts both D_1 and the strict transform of X are nonsingular, so we have no problems there. We will assume now that $k \geq 4$ and we will see later what happens if $k = 2, 3$.

Step 2: Let us first get rid of the A_1 singularity. Thus we blow up in the origin of chart (2). Since this blow-up is an isomorphism outside this point, we preserve the other coordinate charts and we replace chart (2) by the following charts:

$$\begin{cases} x_1^4 x_2^2 (x_1^{4k-6} x_2^{2k-3} + x_2 + x_3^2 + \cdots + x_m^2) = 0 & (2.1) \\ x_2^4 (x_1^{2k-1} x_2^{4k-6} + x_1 + x_3^2 + \cdots + x_m^2) = 0 & (2.2) \\ x_2^2 x_3^4 (x_1^{2k-1} x_2^{2k-3} x_3^{4k-6} + x_1 x_2 + 1 + x_4^2 + \cdots + x_m^2) = 0 & (2.3) \\ \quad \vdots & \vdots \\ x_2^2 x_m^4 (x_1^{2k-1} x_2^{2k-3} x_m^{4k-6} + x_1 x_2 + x_3^2 + \cdots + x_{m-1}^2 + 1) = 0. & (2.m) \end{cases}$$

Now we see that the strict transform $\widetilde{X}$ of X is nonsingular in this part, but we still have the singular line on D_1 (in charts (1) and (2.1) now). Our new exceptional divisor, we call it E_1, is globally a nonsingular quadric in $\mathbb{P}^{m-1}$.

We check immediately that D_1 and E_1 intersect transversally outside the singular line of D_1: take a point $P = (0, 0, \alpha_3, \ldots, \alpha_m)$ on their intersection in chart (2.1) for example (thus $\alpha_3^2 + \cdots + \alpha_m^2 = 0$). We assume that P does not lie on the singular line on D_1 (so at least one of the α_i is nonzero), since we will blow it up later. The local ring $\mathcal{O}_{P,\widetilde{X}}$ is isomorphic to $\left(\frac{\mathbb{C}[x_1,\ldots,x_m]}{I}\right)_{m_P}$ with $I = (x_1^{4k-6} x_2^{2k-3} + x_2 + x_3^2 + \cdots + x_m^2)$ and $m_P = \frac{(x_1, x_2, x_3-\alpha_3, \ldots, x_m-\alpha_m)}{I}$. As a $\mathbb{C}$-vector space, $\frac{m_P}{m_P^2}$ has dimension $m-1$ and is isomorphic to $\frac{(x_1, x_2, x_3-\alpha_3, \ldots, x_m-\alpha_m)}{(x_1^2, x_1x_2, x_2^2, x_3^2-2\alpha_3x_3+\alpha_3^2, \ldots)+I}$. It is generated by the set $\{x_1, x_2, x_3-\alpha_3, \ldots, x_m-\alpha_m\}$ and the last $m-1$ generators are linearly dependent, since

$$\begin{aligned} & x_2 + 2\alpha_3(x_3 - \alpha_3) + \cdots + 2\alpha_m(x_m - \alpha_m) \\ & \quad = \quad x_2 + 2\alpha_3 x_3 + \cdots + 2\alpha_m x_m \\ & \quad = \quad x_1^{4k-6} x_2^{2k-3} + x_2 + x_3^2 + \cdots + x_m^2 - (x_1^{4k-7} x_2^{2k-4}) x_1 x_2 \\ & \qquad\quad -(x_3^2 - 2\alpha_3 x_3 + \alpha_3^2) - \cdots - (x_m^2 - 2\alpha_m x_m + \alpha_m^2) \\ & \quad = \quad 0, \end{aligned}$$

and thus x_1 and x_2 must be linearly independent. Hence D_1 and E_1 have normal crossings at $(0, 0, \alpha_3, \ldots, \alpha_m)$. Later on, we will not check the normal crossings condition any more, it will be satisfied for all divisors in the end.

Step 3: We tackle the D_{n-2} singularity in chart (1) now. We blow up in its origin:

$$\begin{cases} x_1^4(x_1^{2k-5}+x_1x_2^2+x_3^2+\cdots+x_m^2)=0 & (1.1)\\ x_1^2x_2^4(x_1^{2k-3}x_2^{2k-5}+x_1x_2+x_3^2+\cdots+x_m^2)=0 & (1.2)\\ x_1^2x_3^4(x_1^{2k-3}x_3^{2k-5}+x_1x_2^2x_3+1+x_4^2+\cdots+x_m^2)=0 & (1.3)\\ \quad\vdots & \vdots\\ x_1^2x_m^4(x_1^{2k-3}x_m^{2k-5}+x_1x_2^2x_m+x_3^2+\cdots+x_{m-1}^2+1)=0. & (1.m)\end{cases}$$

It is no surprise that we find a D_{n-4} singularity in the origin of chart (1.1) and an A_1 in the origin of chart (1.2). The newly created divisor, called D_2, intersects D_1 and has a singular line in charts (1.1) and (1.2); the singular line of D_1 from chart (1) is transferred to chart (1.2).

Step 4: We blow up in the origin of chart (1.2). The singularity is resolved and the new divisor E_2 intersects both D_1 and D_2:

$$\begin{cases} x_1^8x_2^4(x_1^{4k-10}x_2^{2k-5}+x_2+x_3^2+\cdots+x_m^2)=0 & (1.2.1)\\ x_1^2x_2^8(x_1^{2k-3}x_2^{4k-10}+x_1+x_3^2+\cdots+x_m^2)=0 & (1.2.2)\\ x_1^2x_2^4x_3^8(x_1^{2k-3}x_2^{2k-5}x_3^{4k-10}+x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (1.2.3)\\ \quad\vdots & \vdots\\ x_1^2x_2^4x_m^8(x_1^{2k-3}x_2^{2k-5}x_m^{4k-10}+x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0. & (1.2.m)\end{cases}$$

The singular lines on D_1 and D_2 are separated and go to charts (1.2.2) and (1.2.1) respectively.

We continue in this way, performing alternate blow-ups in a D_i and an A_1, until we have to blow up in a D_4 singularity.

Step $n-3$: We blow up in the origin of the chart $x_1^{2k-4}(x_1^3+x_1x_2^2+x_3^2+\cdots+x_m^2)=0$.

$$\begin{cases} x_1^{2k-2}(x_1+x_1x_2^2+x_3^2+\cdots+x_m^2)=0 & (1')\\ x_1^{2k-4}x_2^{2k-2}(x_1^3x_2+x_1x_2+x_3^2+\cdots+x_m^2)=0 & (2')\\ x_1^{2k-4}x_3^{2k-2}(x_1^3x_3+x_1x_2^2x_3+1+x_4^2+\cdots+x_m^2)=0 & (3')\\ \quad\vdots & \vdots\\ x_1^{2k-4}x_m^{2k-2}(x_1^3x_m+x_1x_2^2x_m+x_3^2+\cdots+x_{m-1}^2+1)=0. & (m')\end{cases}$$

In fact (j') stands here for $(\underbrace{1.1\ldots1}_{k-2\text{ times}}.j)$. We get three singular points, all analytically isomorphic to an A_1 singularity. Both present divisors (we call them of course D_{k-2} and D_{k-1}) have a singular line and in fact all the singular points lie on the singular line of D_{k-1}. One of the singular points, the origin of chart $(2')$, lies

on the intersection of D_{k-2} and D_{k-1}. Note that the singular points $(0,i,0,\dots,0)$ and $(0,-i,0,\dots,0)$ of chart $(1')$ correspond to the points $(-i,0,\dots,0)$ and $(i,0,\dots,0)$ of chart $(2')$ respectively.

Step $n-2$: We deal with the origin of chart $(2')$ first. Blowing it up yields a divisor E_{k-1} that intersects D_{k-1} and D_{k-2}:

$$\begin{cases} x_1^{4k-4}x_2^{2k-2}(x_1^2x_2+x_2+x_3^2+\cdots+x_m^2)=0 & (2'.1)\\ x_1^{2k-4}x_2^{4k-4}(x_1^3x_2^2+x_1+x_3^2+\cdots+x_m^2)=0 & (2'.2)\\ x_1^{2k-4}x_2^{2k-2}x_3^{4k-4}(x_1^3x_2x_3^2+x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (2'.3)\\ \quad\vdots & \vdots\\ x_1^{2k-4}x_2^{2k-2}x_m^{4k-4}(x_1^3x_2x_m^2+x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0. & (2'.m) \end{cases}$$

The other two singularities lie in charts $(1')$ and $(2'.1)$. The singular lines on D_{k-2} and D_{k-1} get separated and go to charts $(2'.2)$ and $(2'.1)$, respectively.

Step $n-1$: After a coordinate transformation the equation of chart $(1')$ becomes $x_1^{2k-2}(x_1x_2(x_2+2i)+x_3^2+\cdots+x_m^2=0$. To put the same point in the origin, we have to change the equation of chart $(2'.1)$ to $(x_1-i)^{4k-4}x_2^{2k-2}(x_1x_2(x_1-2i)+x_3^2+\cdots+x_m^2)=0$ for example. In this step we blow up both charts in the origin and we call the new divisor F_1:

$$\begin{cases} x_1^{2k}(x_2(x_1x_2+2i)+x_3^2+\cdots+x_m^2)=0 & (1'.1)\\ x_1^{2k-2}x_2^{2k}(x_1(x_2+2i)+x_3^2+\cdots+x_m^2)=0 & (1'.2)\\ x_1^{2k-2}x_3^{2k}(x_1x_2(x_2x_3+2i)+1+x_4^2+\cdots+x_m^2)=0 & (1'.3)\\ \quad\vdots & \vdots\\ x_1^{2k-2}x_m^{2k}(x_1x_2(x_2x_m+2i)+x_3^2+\cdots+x_{m-1}^2+1)=0 & (1'.m) \end{cases} \quad \text{and}$$

$$\begin{cases} x_1^{2k}(x_1-i)^{4k-4}x_2^{2k-2}(x_2(x_1-2i)+x_3^2+\cdots+x_m^2)=0 & (2'.1.1)\\ (x_1x_2-i)^{4k-4}x_2^{2k}(x_1(x_1x_2-2i)+x_3^2+\cdots+x_m^2)=0 & (2'.1.2)\\ (x_1x_3-i)^{4k-4}x_2^{2k-2}x_3^{2k}(x_1x_2(x_1x_3-2i)+1+\cdots+x_m^2)=0 & (2'.1.3)\\ \quad\vdots & \vdots\\ (x_1x_m-i)^{4k-4}x_2^{2k-2}x_m^{2k}(x_1x_2(x_1x_m-2i)+x_3^2+\cdots+1)=0. & (2'.1.m) \end{cases}$$

The last singular point and the singular line on D_{k-1} are now in charts $(1'.2)$ and $(2'.1.1)$.

Step n: Before blowing up the final singular point, we first do a coordinate transformation in chart $(1'.2)$ to get the equation $x_1^{2k-2}(x_2-2i)^{2k}(x_1x_2+x_3^2+\cdots+x_m^2)=0$ and in chart $(2'.1.1)$ to get $(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}(x_1x_2+x_3^2+$

$\cdots + x_m^2) = 0$. The new exceptional divisor is called F_2.

$$\begin{cases} x_1^{2k}(x_1x_2-2i)^{2k}(x_2+x_3^2+\cdots+x_m^2)=0 & (1'.2.1)\\ x_1^{2k-2}(x_2-2i)^{2k}x_2^{2k}(x_1+x_3^2+\cdots+x_m^2)=0 & (1'.2.2)\\ x_1^{2k-2}(x_2x_3-2i)^{2k}x_3^{2k}(x_1x_2+1+x_4^2+\cdots+x_m^2)=0 & (1'.2.3)\\ \qquad\vdots & \vdots \\ x_1^{2k-2}(x_2x_m-2i)^{2k}x_m^{2k}(x_1x_2+x_3^2+\cdots+x_{m-1}^2+1)=0 & (1'.2.m)\end{cases} \quad \text{and}$$

$$\begin{cases} x_1^{2k}(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}(x_2+x_3^2+\cdots+x_m^2)=0 & (2'.1.1.1)\\ (x_1x_2+2i)^{2k}(x_1x_2+i)^{4k-4}x_2^{2k}(x_1+x_3^2+\cdots+x_m^2)=0 & (2'.1.1.2)\\ (x_1x_3+2i)^{2k}(x_1x_3+i)^{4k-4}x_2^{2k-2}x_3^{2k}(x_1x_2+1+\cdots+x_m^2)=0 & (2'.1.1.3)\\ \qquad\vdots & \vdots\\ (x_1x_m+2i)^{2k}(x_1x_m+i)^{4k-4}x_2^{2k-2}x_m^{2k}(x_1x_2+x_3^2+\cdots+1)=0. & (2'.1.1.m)\end{cases}$$

The singular line on D_{k-1} is moved to charts $(1'.2.2)$ and $(2'.1.1.1)$.

In the next $k-1$ steps we blow up in the singular lines on the divisors D_i. This gives rise to new exceptional divisors which will be denoted by G_i. After $k-1$ steps we finally have a log resolution; we will perform steps $n+1$ and $n+k-1$ explicitly.

Step $n+1$: To cover the singular line on D_1 completely, we have to perform the blow-up in charts (2.1) and (1.2.2). In chart (2.1) we have to blow up the variety $Y = \{x_1^4x_2^2(x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2)=0\} \subset \mathbb{A}^m$ in the line $\{x_2=\cdots=x_m=0\}$. The strict transform of Y and the exceptional locus form a reducible variety in $\mathbb{A}^m \times \mathbb{P}^{m-2}$, given by the equations

$$\begin{cases} x_1^4x_2^2(x_1^{4k-6}x_2^{2k-3}+x_2+x_3^2+\cdots+x_m^2)=0 & \\ x_iz_j = x_jz_i & \forall\, i,j\in\{2,\ldots,m\},\end{cases}$$

where $(z_2,\ldots,z_m)$ are homogenous coordinates on $\mathbb{P}^{m-2}$. As for a point blow-up, we can replace x_j by $x_i\frac{z_j}{z_i}$ in the open set $z_i\neq 0$ and rename $\frac{z_j}{z_i}$ as x_j. Hence we get the following equations for Y':

$$\begin{cases} x_1^4x_2^3(x_1^{4k-6}x_2^{2k-4}+1+x_2x_3^2+\cdots+x_2x_m^2)=0 & (2.1.2)\\ x_1^4x_2^2x_3^3(x_1^{4k-6}x_2^{2k-3}x_3^{2k-4}+x_2+x_3+x_3x_4^2+\cdots+x_3x_m^2)=0 & (2.1.3)\\ \qquad\vdots & \vdots\\ x_1^4x_2^2x_m^3(x_1^{4k-6}x_2^{2k-3}x_m^{2k-4}+x_2+x_3^2x_m+\cdots+x_{m-1}^2x_m+x_m)=0. & (2.1.m)\end{cases}$$

The equations after blowing up in $\{x_1=x_3=\cdots=x_m=0\}$ in chart (1.2.2) are:

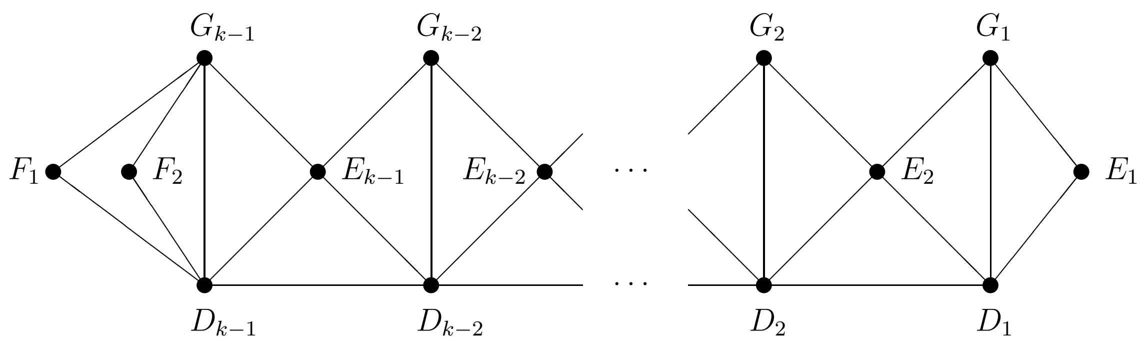
$$\begin{cases} x_1^3x_2^8(x_1^{2k-4}x_2^{4k-10}+1+x_1x_3^2+\cdots+x_1x_m^2)=0 & (1.2.2.1)\\ x_1^2x_2^8x_3^3(x_1^{2k-3}x_2^{4k-10}x_3^{2k-4}+x_1+x_3+x_3x_4^2+\cdots+x_3x_m^2)=0 & (1.2.2.3)\\ \qquad\vdots & \vdots\\ x_1^2x_2^8x_m^3(x_1^{2k-3}x_2^{4k-10}x_m^{2k-4}+x_1+x_3^2x_m+\cdots+x_{m-1}^2x_m+x_m)=0. & (1.2.2.m)\end{cases}$$

Step $n+k-1$: Here we have to consider charts $(1'.2.2)$ and $(2'.1.1.1)$ in which D_{k-1} still has a singular line with equations $\{x_1 = x_3 = \cdots = x_m = 0\}$ and $\{x_2 = x_3 = \cdots = x_m = 0\}$, respectively. Blowing it up yields

$$\left\{\begin{array}{lr} x_1^{2k-1}(x_2-2i)^{2k}x_2^{2k}(1+x_1x_3^2+\cdots+x_1x_m^2)=0 & (1'.2.2.1)\\ x_1^{2k-2}(x_2-2i)^{2k}x_2^{2k}x_3^{2k-1}(x_1+x_3+\cdots+x_3x_m^2)=0 & (1'.2.2.3)\\ \vdots & \vdots \\ x_1^{2k-2}(x_2-2i)^{2k}x_2^{2k}x_m^{2k-1}(x_1+x_3^2x_m+\cdots+x_m)=0 & (1'.2.2.m)\end{array}\right. \quad \text{and}$$

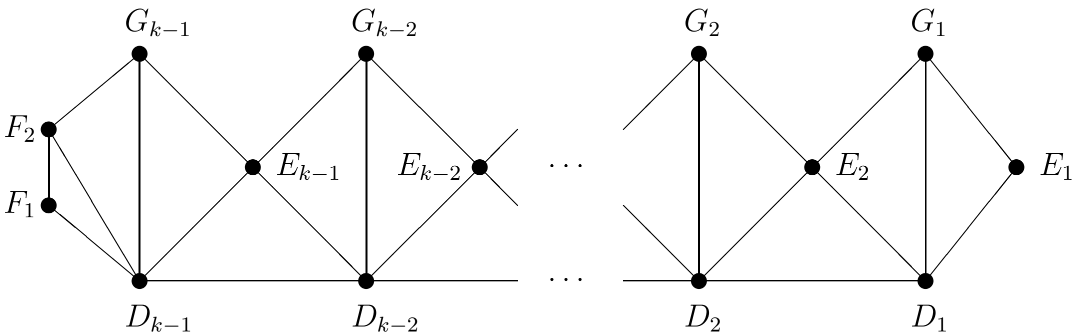
$$\left\{\begin{array}{lr} x_1^{2k}(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-1}(1+x_2x_3^2+\cdots+x_2x_m^2)=0 & (2'.1.1.1.2)\\ x_1^{2k}(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}x_3^{2k-1}(x_2+x_3+\cdots+x_3x_m^2)=0 & (2'.1.1.1.3)\\ \vdots & \vdots \\ x_1^{2k}(x_1+2i)^{2k}(x_1+i)^{4k-4}x_2^{2k-2}x_m^{2k-1}(x_2+x_3^2x_m+\cdots+x_m)=0. & (2'.1.1.1.m)\end{array}\right.$$

From these calculations, we can deduce the intersection diagram. We leave it to the reader to check the details. It can be easily seen that the same diagram is valid for $k = 2, 3$.



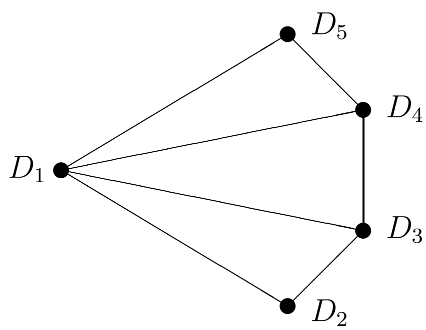
<u>(2) n odd, $n = 2k+1$, with $k \geq 2$.</u>

The first $2k-4$ steps are completely analogous to the case where n is even. Now we end up with the equation $x_1^{2k-4}(x_1^4 + x_1x_2^2 + x_3^2 + \cdots + x_m^2)$ which has a D_5 singularity in the origin. Blowing this up gives one A_3 singularity on the new divisor D_{k-1} (the equation of the first chart is $x_1^{2k-2}(x_1^2+x_1x_2^2+x_3^2+\cdots+x_m^2 = 0)$). We already know that this can be resolved by two consecutive blow-ups, creating divisors F_1 and F_2. Afterwards, the singular lines on the D_i must be blown up. Explicit calculations will lead to the following intersection diagram:



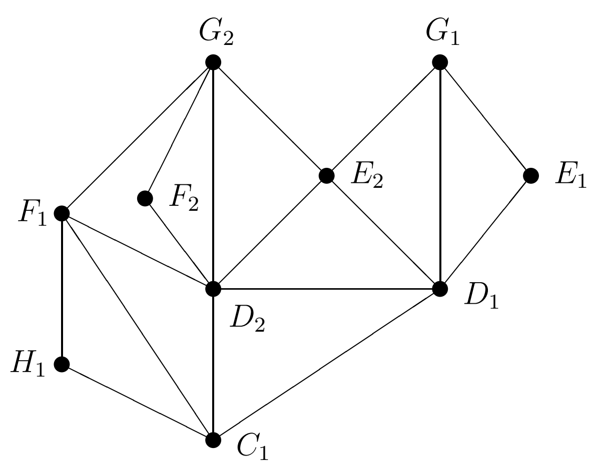
2.5. $\boxed{\textbf{Case } \mathbf{E_6}}$

After blowing up in the origin we get an A_5 singularity and a singular line on the first exceptional divisor D_1. To resolve the A_5 singularity we need three more point blow-ups (creating D_2, D_3 and D_4) and in the end we blow up in the singular line (giving rise to a divisor D_5). We find as intersection graph:

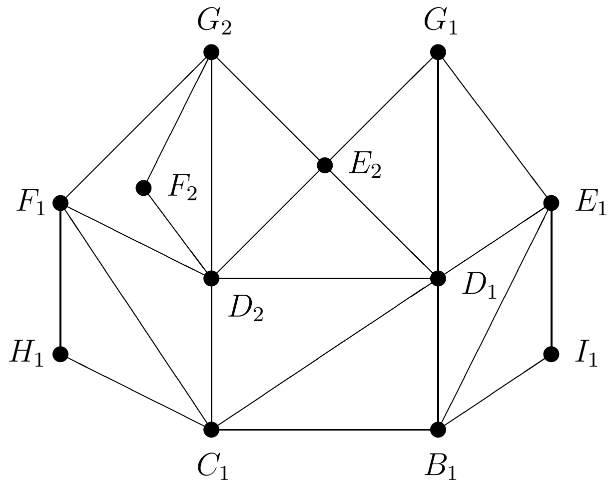


2.6. $\boxed{\textbf{Cases } \mathbf{E_7} \textbf{ and } \mathbf{E_8}}$

An E_7 becomes a D_6 after one step and calculating the intersections gives the following diagram



where C_1 is the very first exceptional divisor and where H_1 arises after blowing up the singular line on C_1. The other divisors come from the D_6 singularity. Notice the difference between F_1 and F_2. It is easy to see that an E_8 singularity passes to an E_7 after one blow-up, with again a singular line on the first exceptional divisor B_1. We denote the divisor that appears after blowing up in this singular line by I_1 and we find the following intersection graph:



3 The Hodge-Deligne polynomials of the pieces of the exceptional locus

3.1. Denote by a_r, b_r, c_r $(r \geq 2)$ the Hodge-Deligne polynomials of

- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r+1}$,
- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r}$,
- $\{x_1^2 + \cdots + x_r^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^{r-1}$,

respectively, where $\mathbb{P}^s$ gets coordinates $(x_1, \ldots, x_{s+1})$. We will be able to express all the needed Hodge-Deligne polynomials in terms of a_r, b_r and c_r, and these last expressions are well known. For completeness we include their computation in the following lemma. *From now on, we will write w as abbreviation of uv.*

Lemma. *The formulae for a_r, b_r and c_r are given in the following table*:

	r *even*	r *odd*
a_r	$\frac{w^{r+1}-1}{w-1}+w^{\frac{r}{2}+1}$	$\frac{w^{r+1}-1}{w-1}$
b_r	$\frac{w^{r}-1}{w-1}+w^{\frac{r}{2}}$	$\frac{w^{r}-1}{w-1}$
c_r	$\frac{w^{r-1}-1}{w-1}+w^{\frac{r}{2}-1}$	$\frac{w^{r-1}-1}{w-1}$

Proof: Denote by d_r the Hodge-Deligne polynomial of $\{x_1^2+\cdots+x_r^2+1=0\}\subset\mathbb{A}^r$. First we compute d_r by induction on r. Since d_2 is the Hodge-Deligne polynomial of a conic with two points at infinity, it equals $w-1$. The variety $\{x_1^2+x_2^2+x_3^2+1=0\}\subset\mathbb{A}^3$ can be regarded as $\mathbb{P}^1\times\mathbb{P}^1$ minus a conic and thus $d_3=(w+1)^2-(w+1)=w^2+w$. For $r\geq 4$ we use the isomorphism $\{x_1^2+\cdots+x_r^2+1=0\}\cong\{x_1x_2+x_3^2+\cdots+x_r^2+1=0\}$. If $x_1=0$ in this last equation, then the contribution to d_r is wd_{r-2} and if $x_1\neq 0$, then it is $(w-1)w^{r-2}$, so we have the recursion formula $d_r=wd_{r-2}+(w-1)w^{r-2}$. From this it follows that $d_r=w^{r-1}-w^{\frac{r}{2}-1}$ if r is even and $d_r=w^{r-1}+w^{\frac{r-1}{2}}$ if r is odd.

For a_2 we find $2w^2+w+1$ and we have the recursion formula $a_r=a_{r-1}+w^2d_{r-1}$ for $r\geq 3$. The formulae for b_r and c_r can be deduced similarly. ∎

3.2. For the remainder of this section, we will calculate the Hodge-Deligne polynomials of the pieces D_J° (see the definition of the stringy E-function). Since we are mainly interested in the contribution of the singular point (by which we mean $E_{st}(X)-H(D_\emptyset^\circ)=E_{st}(X)-H(X\setminus\{0\})$, where X is a defining variety of an A-D-E singularity), we will do this for $J\neq\emptyset$.

We remark here the following. In the defining formula of the stringy E-function we need the Hodge-Deligne polynomials of the D_J° at the end of the resolution process. Notice however that we can compute them immediately after they are created, since a blow-up is an isomorphism outside its center. So we just have to subtract contributions of intersections with previously created divisors and already present centers of future blow-ups from the global Hodge-Deligne polynomial in the right way.

The case of an A-D-E surface singularity is well known and for threefold singularities we refer again to [DR], so we consider here the higher dimensional case. Parallel to the previous section, we will work out the details for the case D_n, n even, and state the results in the other cases. We use the same notations as in the previous section.

3.3. $\boxed{\textbf{Case A}}$

From the description in (2.3), one gets the following:

(1) n odd

$$
\begin{array}{ll}
H(D_1^\circ) = b_{m-1} - 1 & \\
H(D_i^\circ) = b_{m-1} - c_{m-1} - 1 & (i = 2, \dots, k-1) \\
H(D_k^\circ) = c_m - c_{m-1} & \\
H(D_i \cap D_{i+1}) = c_{m-1} & (i = 1, \dots, k-1)
\end{array}
$$

(2) n even

$$
\begin{array}{ll}
H(D_1^\circ) = b_{m-1} - 1 & \\
H(D_i^\circ) = b_{m-1} - c_{m-1} - 1 & (i = 2, \dots, k) \\
H(D_{k+1}^\circ) = w^{m-2} + \cdots + 1 - c_{m-1} & \\
H(D_i \cap D_{i+1}) = c_{m-1} & (i = 1, \dots, k)
\end{array}
$$

3.4. $\boxed{\textbf{Case D}}$

(1) n even

All the needed information can be read off from the equations in (2.4). We follow the same steps.

Step 1: The first exceptional divisor is globally isomorphic to $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, which has a singular line that contains the two singular points of the surrounding variety. Hence $H(D_1^\circ) = a_{m-2} - (w+1)$.

Step 2: One sees that E_1 is a nonsingular quadric in $\mathbb{P}^{m-1}$ that intersects D_1 in $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$, for a suitable choice of coordinates. Thus $H(E_1^\circ) = c_m - b_{m-2}$. The intersection of D_1 and E_1 contains one point of the singular line on D_1 and hence $H((D_1 \cap E_1)^\circ) = b_{m-2} - 1$.

Step 3: Analogous to step 1 one finds that D_2 is isomorphic to $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-1}$, with a singular line that contains two singular points of the surrounding variety. Now D_2 intersects D_1 in $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$. This intersection has exactly one point (the origin of coordinate chart (1.2)) in common with the singular lines on D_2 and D_1. The conclusion is that $H(D_2^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$ and $H((D_1 \cap D_2)^\circ) = b_{m-2} - 1$.

Step 4: For $H(E_2^\circ)$ we find $c_m - 2b_{m-2} + c_{m-2}$, where $2b_{m-2}$ comes from the intersections with D_1 and D_2 and c_{m-2} from the intersection with $D_1 \cap D_2$. We also

have that $H((D_1 \cap E_2)^\circ) = H((D_2 \cap E_2)^\circ) = b_{m-2} - c_{m-2} - 1$, where the -1 comes from a point on the singular lines on the D_i. Finally $H(D_1 \cap D_2 \cap E_2) = c_{m-2}$.

Analogously, for all i from 3 to $k-2$, we have $H(D_i^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$, $H((D_{i-1} \cap D_i)^\circ) = b_{m-2} - 1$, $H(E_i^\circ) = c_m - 2b_{m-2} + c_{m-2}$, $H((D_{i-1} \cap E_i)^\circ) = H((D_i \cap E_i)^\circ) = b_{m-2} - c_{m-2} - 1$ and $H(D_{i-1} \cap D_i \cap E_i) = c_{m-2}$.

Step $n-3$: In this step three singular points are created, but since they are all on the singular line on D_{k-1}, we still find $H(D_{k-1}^\circ) = a_{m-2} - (w+1) - b_{m-2} + 1$ and $H((D_{k-2} \cap D_{k-1})^\circ) = b_{m-2} - 1$.

Step $n-2$: Again nothing special happens: $H(E_{k-1}^\circ) = c_m - 2b_{m-2} + c_{m-2}$, $H((D_{k-2} \cap E_{k-1})^\circ) = H((D_{k-1} \cap E_{k-1})^\circ) = b_{m-2} - c_{m-2} - 1$ and $H(D_{k-2} \cap D_{k-1} \cap E_{k-1}) = c_{m-2}$.

Step $n-1$ and step n: Both F_1 and F_2 are nonsingular quadrics in $\mathbb{P}^{m-1}$ and their intersection with D_{k-1} is $\{x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{P}^{m-2}$, which has one point in common with the singular line on D_{k-1}. Thus $H(F_1^\circ) = H(F_2^\circ) = c_m - b_{m-2}$ and $H((D_{k-1} \cap F_1)^\circ) = H((D_{k-1} \cap F_2)^\circ) = b_{m-2} - 1$.

Step $n+1$: The singular line on D_1 is except for the origin of coordinate chart (2.1) covered by chart (1.2.2). But after the blow-up, exactly the intersection of E_1 and G_1 lies above the origin of chart (2.1). Thus to calculate $H(G_1^\circ)$, it suffices to consider only charts (1.2.2.1) to (1.2.2.m). In chart (1.2.2.3) G_1 is just isomorphic to $\mathbb{A}^{m-2}$. The piece of G_1 that is covered by chart (1.2.2.4) but not by (1.2.2.3) is isomorphic to $\mathbb{A}^{m-3}$ and so on, until we add an affine line to G_1 in chart (1.2.2.m). The intersection of G_1 with E_2 is isomorphic to $\mathbb{P}^{m-3}$. It is not so hard to see that $H(D_1 \cap E_2 \cap G_1) = c_{m-2}$ (notice that the equations of (the strict transform of) D_1 in chart (1.2.2.3) for instance are $x_1 = 0$ and $1 + x_4^2 + \cdots + x_m^2 = 0$), and from this it follows that $H((D_1 \cap G_1)^\circ) = (w-1)c_{m-2}$ (the w comes from the x_2-coordinate that can be chosen freely in every chart). Now we also have $H((E_2 \cap G_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and $H(G_1^\circ) = w^{m-2} + \cdots + w - (w^{m-3} + \cdots + 1) - wc_{m-2} + c_{m-2} = w^{m-2} - 1 - (w-1)c_{m-2}$. One gets from charts (2.1.2) to (2.1.m) that $H((E_1 \cap G_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and that $H(D_1 \cap E_1 \cap G_1) = c_{m-2}$.

More conceptually, G_1 is a locally trivial $\mathbb{P}^{m-3}$-bundle over the singular line on D_1 and $E_1 \cap G_1$ and $E_2 \cap G_1$ are two fibers. Thus $H(G_1) = (w+1)(w^{m-3} + \cdots + 1)$ and $H(E_i \cap G_1) = w^{m-3} + \cdots + 1$. Furthermore, we can consider the singular line on D_1 as a family of A_1 singularities and thus $D_1 \cap G_1$ is a family of nonsingular quadrics in $\mathbb{P}^{m-3}$. This implies that $H(D_1 \cap G_1) = (w+1)c_{m-2}$ and $H(D_1 \cap E_i \cap G_1) = c_{m-2}$.

In exactly the same way one finds that (for $i \in \{2, \dots, k-2\}$) $H(G_i^\circ) = w^{m-2} - 1 - (w-1)c_{m-2}$, $H((D_i \cap G_i)^\circ) = (w-1)c_{m-2}$, $H((E_i \cap G_i)^\circ) = H((E_{i+1} \cap G_i)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$ and $H(D_i \cap E_i \cap G_i) = H(D_i \cap E_{i+1} \cap G_i) = c_{m-2}$.

Step $n + k - 1$: This step looks very much like step $n + 1$. It suffices to consider charts $(1'.2.2.1)$ to $(1'.2.2.m)$ to compute $H(G_{k-1}^\circ)$. One checks that $H(D_{k-1} \cap F_1 \cap G_{k-1}) = H(D_{k-1} \cap F_2 \cap G_{k-1}) = c_{m-2}$, $H((F_1 \cap G_{k-1})^\circ) = H((F_2 \cap G_{k-1})^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$, $H((D_{k-1} \cap G_{k-1})^\circ) = (w-2)c_{m-2}$ and thus $H(G_{k-1}^\circ) = w^{m-2} + \cdots + w - 2(w^{m-3} + \cdots + 1) - (w-2)c_{m-2}$. From charts $(2'.1.1.1.2)$ to $(2'.1.1.1.m)$ we get $H(D_{k-1} \cap E_{k-1} \cap G_{k-1}) = c_{m-2}$ and $H((E_{k-1} \cap G_{k-1})^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}$.

A conceptual explanation like in step $n + 1$ can be given here too.

(2) n odd

There are only 7 changes in comparison with the case where n is even. First remark that $F_1 \cap G_{k-1}$ and $D_{k-1} \cap F_1 \cap G_{k-1}$ are empty, but instead $H((F_1 \cap F_2)^\circ) = c_{m-1} - c_{m-2}$ and $H(D_{k-1} \cap F_1 \cap F_2) = c_{m-2}$. The other 5 changes are the following:

$$
\begin{aligned}
&H(F_1^\circ) = b_{m-1} - b_{m-2} \\
&H(F_2^\circ) = c_m - c_{m-1} - b_{m-2} + c_{m-2} \\
&H(G_{k-1}^\circ) = w^{m-2} - 1 - (w-1)c_{m-2} \\
&H((D_{k-1} \cap F_2)^\circ) = b_{m-2} - c_{m-2} - 1 \\
&H((D_{k-1} \cap G_{k-1})^\circ) = (w-1)c_{m-2}
\end{aligned}
$$

3.5. $\boxed{\textbf{Case } \mathbf{E_6}}$

We just list the results.

$$
\begin{aligned}
&H(D_1^\circ) = a_{m-2} - w - 1 \\
&H(D_2^\circ) = b_{m-1} - b_{m-2} \\
&H(D_3^\circ) = b_{m-1} - b_{m-2} - c_{m-1} + c_{m-2} \\
&H(D_4^\circ) = c_m - b_{m-2} - c_{m-1} + c_{m-2} \\
&H(D_5^\circ) = w^{m-2} + \cdots + w - wc_{m-2} \\
&H((D_1 \cap D_2)^\circ) = b_{m-2} - 1 \\
&H((D_1 \cap D_3)^\circ) = H((D_1 \cap D_4)^\circ) = b_{m-2} - c_{m-2} - 1 \\
&H((D_1 \cap D_5)^\circ) = wc_{m-2} \\
&H((D_2 \cap D_3)^\circ) = H((D_3 \cap D_4)^\circ) = c_{m-1} - c_{m-2} \\
&H((D_4 \cap D_5)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2} \\
&H(D_1 \cap D_2 \cap D_3) = H(D_1 \cap D_3 \cap D_4) = H(D_1 \cap D_4 \cap D_5) = c_{m-2}
\end{aligned}
$$

3.6. $\boxed{\textbf{Cases } \mathbf{E_7} \textbf{ and } \mathbf{E_8}}$

Let us first treat the E_8 case. From the intersection diagram it follows that we have to compute 47 Hodge-Deligne polynomials (there are 12 divisors, 23 intersections of 2 divisors and 12 intersections of 3 divisors). But there are 20 polynomials coming from the 'D_6 part' of the diagram that are left unchanged here. So we will only write down the other 27.

$$
\begin{aligned}
&H(B_1^\circ) = a_{m-2} - w - 1\\
&H(C_1^\circ) = a_{m-2} - b_{m-2} - w\\
&H(D_1^\circ) = H(D_2^\circ) = a_{m-2} - 2b_{m-2} + c_{m-2} - w + 1\\
&H(E_1^\circ) = H(F_1^\circ) = c_m - 2b_{m-2} + c_{m-2}\\
&H(H_1^\circ) = H(I_1^\circ) = w^{m-2} + \cdots + w - wc_{m-2}\\
&H((B_1 \cap C_1)^\circ) = H((B_1 \cap I_1)^\circ) = H((C_1 \cap H_1)^\circ) = wc_{m-2}\\
&H((B_1 \cap D_1)^\circ) = H((B_1 \cap E_1)^\circ) = H((C_1 \cap D_1)^\circ)\\
&\quad = H((C_1 \cap D_2)^\circ) = H((C_1 \cap F_1)^\circ) = H((D_1 \cap D_2)^\circ)\\
&\quad = H((D_1 \cap E_1)^\circ) = H((D_2 \cap F_1)^\circ) = b_{m-2} - c_{m-2} - 1\\
&H((E_1 \cap I_1)^\circ) = H((F_1 \cap H_1)^\circ) = w^{m-3} + \cdots + 1 - c_{m-2}\\
&H(B_1 \cap C_1 \cap D_1) = H(B_1 \cap D_1 \cap E_1) = H(B_1 \cap E_1 \cap I_1)\\
&\quad = H(C_1 \cap D_1 \cap D_2) = H(C_1 \cap D_2 \cap F_1) = H(C_1 \cap F_1 \cap H_1) = c_{m-2}
\end{aligned}
$$

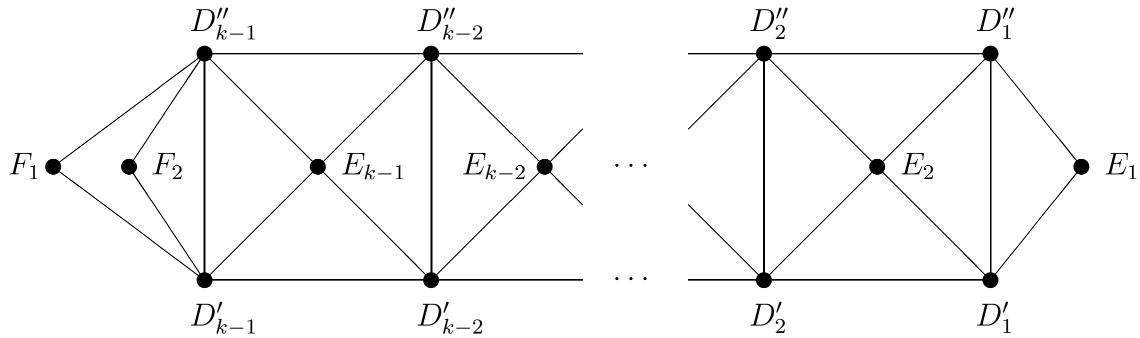
For the E_7 case, we can skip all expressions involving the divisors B_1 and/or I_1. This leaves us with 37 polynomials and apart from the following 5, they are all the same as in the E_8 case.

$$
\begin{aligned}
&H(C_1^\circ) = a_{m-2} - w - 1\\
&H(D_1^\circ) = a_{m-2} - b_{m-2} - w\\
&H(E_1^\circ) = c_m - b_{m-2}\\
&H((C_1 \cap D_1)^\circ) = H((D_1 \cap E_1)^\circ) = b_{m-2} - 1
\end{aligned}
$$

4 Computation of the discrepancy coefficients

4.1. In this section we compute the last data that we need: the discrepancy coefficients. As already mentioned in (1.4), all the two dimensional A-D-E's admit a crepant resolution, this means that all the discrepancies are 0.

For the three-dimensional case, the computations are done in [DR], but the authors are a bit inaccurate. Let us again consider the case D_n, n even, with $k = \frac{n}{2}$. The intersection diagram is as follows:



Compared to the higher dimensional cases, the D_i fall apart into two components D'_i and D''_i, and there are no divisors G_i needed. If we denote by $\varphi : \widetilde{X} \to X$ the log resolution, with X the defining variety of the D_n singularity and $\widetilde{X}$ the strict transform of X, then φ can be decomposed into k birational morphisms

$$\widetilde{X} = X_k \xrightarrow{\varphi_k} X_{k-1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\varphi_2} X_1 \xrightarrow{\varphi_1} X_0 = X,$$

where the exceptional locus of φ_1 is $\{D'_1, D''_1\}$, of φ_i $(2 \le i \le k-1)$ is $\{D'_i, D''_i, E_{i-1}\}$ and of φ_k is $\{F_1, F_2, E_{k-1}\}$, again using the same name for the divisors at any stage of the decomposition of φ. We can also decompose $K_{\widetilde{X}} - \varphi^*(K_X)$ as

$$\left[\sum_{i=1}^{k-1} \varphi_k^*(\varphi_{k-1}^* \cdots (\varphi_{i+1}^*(K_{X_i} - \varphi_i^*(K_{X_{i-1}}))) \cdots)\right] + K_{X_k} - \varphi_k^*(K_{X_{k-1}}).$$

Dais and Roczen calculated that for instance $\varphi_2^*(D'_1) = D'_1 + D'_2 + E_1$ and $\varphi_2^*(D''_1) = D''_1 + D''_2 + E_1$, but D'_1 and D''_1 are not Cartier. Their sum $D'_1 + D''_1$ is Cartier and it turns out that $\varphi_2^*(D'_1 + D''_1) = D'_1 + D''_1 + D'_2 + D''_2 + E_1$ instead of $\cdots + 2E_1$. This kind of error occurs also in the following stages for this type of singularity and also for type D_n, n odd, and for types E_6, E_7 and E_8. In the next table, we list the discrepancies. We use notations analogous to our notations from section 2, but they differ from the notations in [DR]. The coefficients that we have corrected are in boldface.

Type of singularity		Discrepancy
A_n	n even $n = 2k$ $k \ge 1$	$\sum_{i=1}^{k} iD_i + (n+2)D_{k+1}$

	n odd $n = 2k - 1$ $k \geq 1$	$\sum_{i=1}^{k} iD_i$
D_n	n even $n = 2k$ $k \geq 2$	$\sum_{i=1}^{k-1} \bigl(iD_i' + iD_i'' + \mathbf{2i}E_i\bigr) + \mathbf{k}F_1 + \mathbf{k}F_2$
	n odd $n = 2k + 1$ $k \geq 2$	$\sum_{i=1}^{k-1} \bigl(iD_i' + iD_i'' + \mathbf{2i}E_i\bigr) + \mathbf{k}F_1 + \mathbf{2k}F_2$
E_6		$D_1' + D_1'' + \mathbf{2}D_2 + \mathbf{4}D_3 + \mathbf{6}D_4$
E_7		$C_1' + C_1'' + 2D_1' + 2D_1'' + 4D_2' + 4D_2''$ $+\mathbf{3}E_1 + \mathbf{7}E_2 + \mathbf{6}F_1 + \mathbf{5}F_2$
E_8		$B_1' + B_1'' + 2C_1' + 2C_1'' + 4D_1' + 4D_1'' + 7D_2' + 7D_2''$ $+\mathbf{6}E_1 + \mathbf{12}E_2 + \mathbf{10}F_1 + \mathbf{8}F_2$

Remark. Dais and Roczen used their results to contradict a conjecture of Batyrev about the range of the string-theoretic index (see [Ba1, Conjecture 5.9], [DR, Remark 1.9]). Luckily, this follows already from the formulae for the A case, to which we do not correct anything. We will only simplify their formulae in this case.

4.2. Now we consider the higher dimensional case. As an example, we will calculate the discrepancy coefficient of the divisor E_i for an $(m-1)$-dimensional D_n singularity, where n is even, $i \in \{1, \dots, k-1\}$ and $m \geq 5$. Let X be the defining variety $\{x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 = 0\} \subset \mathbb{A}^m$, and let $\varphi : \widetilde{X} \to X$ be the log resolution constructed in section 2. We take a coordinate chart that covers a piece of E_i; in the notation of section 2, this could be for example chart $(\underbrace{1.1\dots1}_{i-1 \text{ times}}.2.3)$ describing an open set $U \subset \widetilde{X}$:

$$y_1^{2k-2i+1} y_2^{2k-2i-1} y_3^{4k-4i-2} + y_1 y_2 + 1 + y_4^2 + \cdots + y_m^2 = 0.$$

In this chart, $y_1 = 0$ gives a local equation for divisor D_{i-1}, $y_2 = 0$ for D_i and $y_3 = 0$ for our divisor E_i. The map $\varphi : U \to X$ can be found from the resolution process. Here it will be

$$\varphi(y_1, \ldots, y_m) = (y_1 y_2 y_3^2, y_1^{i-1} y_2^i y_3^{2i-1}, y_1^{i-1} y_2^i y_3^{2i}, y_1^{i-1} y_2^i y_3^{2i} y_4, \ldots, y_1^{i-1} y_2^i y_3^{2i} y_m).$$

The section $\frac{dx_1 \wedge \ldots \wedge dx_{m-1}}{2x_m}$ is locally a generator of the sheaf $\mathcal{O}_X(K_X)$ ($2x_m = \frac{\partial f}{\partial x_m}$, where f is the equation of X) and we have to compare its pull-back under φ with the generator $\frac{dy_1 \wedge \ldots \wedge dy_{m-1}}{2y_m}$ of $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})|_U$. We have

$$\varphi^*(\frac{dx_1 \wedge \ldots \wedge dx_{m-1}}{2x_m}) = y_1^{(i-1)(m-3)} y_2^{i(m-3)} y_3^{2i(m-3)} \frac{dy_1 \wedge \ldots \wedge dy_{m-1}}{2y_m},$$

which learns us that the discrepancy coefficient of E_i is $2i(m-3)$. And we get the discrepancy coefficient of D_i for free, it is $i(m-3)$. In general, the following can be proven by this kind of calculations.

Proposition. *For all divisors that are created after a point blow-up, except for divisor $D_{\frac{n}{2}+1}$ in the A_n (n even) case, the discrepancy coefficient is $(m-3)$ times the coefficient of the corresponding divisor(s) in the three-dimensional case (see the table in (4.1)).*

What about the other divisors? They are all created after blowing up a nonsingular surrounding variety in a point (case A_n, n even) or a line (other cases). We consider again the case of a D_n singularity, with n even. Denote by $X^{(i)}$ the variety obtained after $n+i$ steps in the resolution process of section 2 ($i \in \{0, \ldots, k-2\}$). The log resolution $\varphi : \widetilde{X} \to X$ can be decomposed as follows:

$$\widetilde{X} \xrightarrow{\chi^{(i+1)}} X^{(i+1)} \xrightarrow{\varphi^{(i+1)}} X^{(i)} \xrightarrow{\psi^{(i)}} X,$$

where $\varphi^{(i+1)}$ is the blow-up of the singular line on the divisor $D_{i+1} \subset X^{(i)}$ and where $\chi^{(i+1)}$ and $\psi^{(i)}$ are compositions of other blow-ups. Notice that all the singular lines on $X^{(0)}$ are disjoint. Thus, to compute the discrepancy coefficient of G_{i+1}, it suffices to look at its coefficient in $K_{X^{(i+1)}} - (\psi^{(i)} \circ \varphi^{(i+1)})^*(K_X)$. This is equal to

$$K_{X^{(i+1)}} - (\varphi^{(i+1)})^*((\psi^{(i)})^*(K_X) - K_{X^{(i)}}) - (\varphi^{(i+1)})^*(K_{X^{(i)}}).$$

It follows from [GH, p.608] that the last term is $-K_{X^{(i+1)}} + (m-3)G_{i+1}$ ($X^{(i)}$ is nonsingular!). And in the second term we only get a nonzero coefficient for G_{i+1} from $-(\varphi^{(i+1)})^*(-(i+1)(m-3)D_{i+1})$ (this follows from [GH, p.605], and the exact coefficient is $2(i+1)(m-3)$ because the multiplicity of a generic point of the singular line on D_{i+1} is 2). This gives us $2(i+1)(m-3)+(m-3) = (2i+3)(m-3)$

as discrepancy coefficient for G_{i+1}. In all other cases where we blow up in a line, the multiplicity of a generic point of the singular line will also be 2 and thus we have the following proposition.

Proposition. *For all divisors that are created after a blow-up in a singular line of another divisor D, the discrepancy coefficient is*

$$2(\textit{discrepancy coefficient of } D) + (m-3).$$

The reader may check that the same arguments give $(n+1)(m-3)+1$ as coefficient for $D_{\frac{n}{2}+1}$ in the case A_n, n even.

5 Formulae for the contribution of an A-D-E singularity to the stringy E-function and application to Batyrev's conjecture

5.1. Let X be a defining variety of an A-D-E singularity; hence X is a hypersurface in $\mathbb{A}^m$ ($m \geq 3$) with a singular point in the origin. By the contribution of the singular point to the stringy E-function, we mean $E_{st}(X) - H(X \setminus \{0\})$ (see (3.2)). Before stating the formulae, we first remark that we have to make a distinction between m even and m odd, because the required Hodge-Deligne polynomials depend on the parity of the dimension.

Theorem. *The contributions of the $(m-1)$-dimensional A-D-E singularities ($m \geq 3$) are given in the following tables (where sums like $\sum_{i=2}^{k}$ must be interpreted as 0 for $k=1$).*

Type of singularity		Contribution of singular point for odd m
A_n	n even $n=2k$ $k \geq 1$	$1 + \frac{(w-1)}{(w^{(2k+1)(m-3)+2}-1)} \left(\sum_{i=2}^{k+1} w^{(k+i)(m-3)+2} + \sum_{i=1}^{k} w^{(k+i)(m-3)+\frac{m+1}{2}} + \sum_{i=1}^{k} w^{i(m-3)+\frac{m-1}{2}} + \sum_{i=1}^{k} w^{i(m-3)+1} \right)$

	n odd $n=2k-1$ $k\geq 1$	$1+\frac{(w-1)}{(w^{k(m-3)+1}-1)}\left(\sum_{i=1}^{k} w^{i(m-3)+1}+\sum_{i=1}^{k-1} w^{i(m-3)+\frac{m-1}{2}}\right)$
D_n	n even $n=2k$ $k\geq 2$	$1+\frac{(w-1)}{(w^{(2k-1)(m-3)+1}-1)}\left(\sum_{i=1}^{2k-1} w^{i(m-3)+1}+w^{k(m-3)+1}\right)$
	n odd $n=2k+1$ $k\geq 2$	$1+\frac{(w-1)}{(w^{2k(m-3)+1}-1)}\left(\sum_{i=1}^{2k} w^{i(m-3)+1}+w^{k(m-3)+\frac{m-1}{2}}\right)$
E_6		$1+\frac{(w-1)}{(w^{6m-17}-1)}\big(w^{6m-17}+w^{4m-11}+w^{3m-8}$ $+w^{m-2}+w^{\frac{9m-25}{2}}+w^{\frac{5m-13}{2}}\big)$
E_7		$1+\frac{(w-1)}{(w^{9m-26}-1)}\big(w^{9m-26}+w^{7m-20}+w^{6m-17}+w^{5m-14}$ $+w^{4m-11}+w^{3m-8}+w^{m-2}\big)$
E_8		$1+\frac{(w-1)}{(w^{15m-44}-1)}\big(w^{15m-44}+w^{12m-35}+w^{10m-29}+w^{9m-26}$ $+w^{7m-20}+w^{6m-17}+w^{4m-11}+w^{m-2}\big)$

Type of singularity		Contribution of singular point for even m
A_n	n even $n=2k$ $k\geq 1$	$1+\frac{(w-1)}{(w^{(2k+1)(m-3)+2}-1)}\left(\sum_{i=2}^{k+1} w^{(k+i)(m-3)+2}+\sum_{i=1}^{k} w^{i(m-3)+1}\right)$

	n odd $n = 2k-1$ $k \geq 1$	$1 + \frac{(w-1)}{(w^{k(m-3)+1}-1)} \left(\sum_{i=1}^{k} w^{i(m-3)+1} + w^{\frac{m}{2}-1} \right)$
D_n	n even $n = 2k$ $k \geq 2$	$1 + \frac{(w-1)}{(w^{(2k-1)(m-3)+1}-1)} \left(\sum_{i=1}^{2k-1} w^{i(m-3)+1} + w^{k(m-3)+1} + \sum_{i=0}^{k-2} w^{(k+i)(m-3)+\frac{m}{2}} + \sum_{i=0}^{k-1} w^{i(m-3)+\frac{m}{2}-1} + w^{\frac{m}{2}-1} \right)$
	n odd $n = 2k+1$ $k \geq 2$	$1 + \frac{(w-1)}{(w^{2k(m-3)+1}-1)} \left(\sum_{i=1}^{2k} w^{i(m-3)+1} + \sum_{i=1}^{k-1} w^{(k+i)(m-3)+\frac{m}{2}} + \sum_{i=0}^{k-1} w^{i(m-3)+\frac{m}{2}-1} \right)$
E_6		$1 + \frac{(w-1)}{(w^{6m-17}-1)} \Big(w^{6m-17} + w^{4m-11} + w^{3m-8} + w^{m-2} + w^{\frac{11m-30}{2}} + w^{\frac{3m-8}{2}} \Big)$
E_7		$1 + \frac{(w-1)}{(w^{9m-26}-1)} \Big(w^{9m-26} + w^{7m-20} + w^{6m-17} + w^{5m-14} + w^{4m-11} + w^{3m-8} + w^{m-2} + w^{\frac{17m-48}{2}} + w^{\frac{15m-42}{2}} + w^{\frac{11m-30}{2}} + w^{\frac{9m-26}{2}} + w^{\frac{5m-14}{2}} + w^{\frac{3m-8}{2}} + w^{\frac{m-2}{2}} \Big)$
E_8		$1 + \frac{(w-1)}{(w^{15m-44}-1)} \Big(w^{15m-44} + w^{12m-35} + w^{10m-29} + w^{9m-26} + w^{7m-20} + w^{6m-17} + w^{4m-11} + w^{m-2} + w^{\frac{29m-84}{2}} + w^{\frac{27m-78}{2}} + w^{\frac{23m-66}{2}} + w^{\frac{17m-48}{2}} + w^{\frac{15m-44}{2}} + w^{\frac{9m-26}{2}} + w^{\frac{5m-14}{2}} + w^{\frac{3m-8}{2}} \Big)$

Proof:

- Let us first consider the case where $m \geq 5$. We will focus again on the singularity of type D_n for $n = 2k$ and also for even m. All the other cases are completely analogous. We just insert the data from sections 2, 3 and 4 in the defining formula of the stringy E-function and we find the following formula for the contribution of the singularity:

$$
\begin{aligned}
&\frac{(w^{m-1} - w^2 + w^{\frac{m+2}{2}} - w^{\frac{m}{2}})}{(w^{m-2}-1)} + \sum_{i=2}^{k-1} \frac{(w^{m-2} - w + w^{\frac{m}{2}} - w^{\frac{m-2}{2}})(w-1)}{(w^{i(m-3)+1}-1)} + \frac{(w^{m-2})(w-1)}{(w^{2m-5}-1)} \\
&+ \sum_{i=2}^{k-1} \frac{(w^{m-2} - w^{m-3} - w^{\frac{m-2}{2}} + w^{\frac{m-4}{2}})(w-1)}{(w^{2i(m-3)+1}-1)} + \frac{2w^{m-2}(w-1)}{(w^{k(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-2} \frac{(w^{m-2} - w^{m-3} - w^{\frac{m-2}{2}} + w^{\frac{m-4}{2}})(w-1)}{(w^{(2i+1)(m-3)+1}-1)} + \frac{(w^{m-2} - 2w^{m-3} - w^{\frac{m-2}{2}} + 2w^{\frac{m-4}{2}})(w-1)}{(w^{(2k-1)(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-2} \frac{(w^{m-2} - w + w^{\frac{m}{2}} - w^{\frac{m-2}{2}})(w-1)}{(w^{i(m-3)+1}-1)(w^{(i+1)(m-3)+1}-1)} + \frac{(w^{m-2} - w + w^{\frac{m}{2}} - w^{\frac{m-2}{2}})(w-1)}{(w^{m-2}-1)(w^{2m-5}-1)} \\
&+ \sum_{i=2}^{k-1} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{2i(m-3)+1}-1)} + \sum_{i=1}^{k-2} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)} \\
&+ \frac{2(w^{m-2} - w + w^{\frac{m}{2}} - w^{\frac{m-2}{2}})(w-1)}{(w^{(k-1)(m-3)+1}-1)(w^{k(m-3)+1}-1)} + \sum_{i=1}^{k-2} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)} \\
&+ \frac{(w^{m-2} - 2w^{m-3} - w + 2 + w^{\frac{m}{2}} - 3w^{\frac{m-2}{2}} + 2w^{\frac{m-4}{2}})(w-1)}{(w^{(k-1)(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-1} \frac{(w^{m-3} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{2i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)} + \sum_{i=1}^{k-2} \frac{(w^{m-3} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{(2i+2)(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)} \\
&+ \frac{2(w^{m-3} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{k(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-2} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(i+1)(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-1} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{2i(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)} \\
&+ \sum_{i=1}^{k-2} \frac{(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{i(m-3)+1}-1)(w^{(2i+2)(m-3)+1}-1)(w^{(2i+1)(m-3)+1}-1)} \\
&+ \frac{2(w^{m-3} - 1 + w^{\frac{m-2}{2}} - w^{\frac{m-4}{2}})(w-1)^2}{(w^{(k-1)(m-3)+1}-1)(w^{k(m-3)+1}-1)(w^{(2k-1)(m-3)+1}-1)}.
\end{aligned}
$$

The terms correspond to the following pieces of the exceptional locus (in that order):

$$
\begin{aligned}
&D_1^\circ, D_i^\circ, E_1^\circ, E_i^\circ, F_i^\circ, G_i^\circ, G_{k-1}^\circ, (D_i \cap D_{i+1})^\circ, (D_1 \cap E_1)^\circ, (D_i \cap E_i)^\circ, \\
&(D_i \cap E_{i+1})^\circ, (D_{k-1} \cap F_i)^\circ, (D_i \cap G_i)^\circ, (D_{k-1} \cap G_{k-1})^\circ, (E_i \cap G_i)^\circ, \\
&(E_{i+1} \cap G_i)^\circ, (F_i \cap G_{k-1})^\circ, D_i \cap D_{i+1} \cap E_{i+1}, D_i \cap E_i \cap G_i, \\
&D_i \cap E_{i+1} \cap G_i, D_{k-1} \cap F_i \cap G_{k-1}.
\end{aligned}
$$

By a very long but easy calculation, it can be proved by induction on k that we indeed get the requested formula. We remark here that we have

done the computations for $m \geq 5$, for $m = 4$ and for $m = 3$ separately, and then noticed that the formulae for $m \geq 5$ are correct in the other cases too.

- We can now explain why these formulae are also valid for $m = 4$. For the A_n case, this is not a surprise, since the intersection diagram for $m = 4$ is the same as for $m \geq 5$.

  For the other cases, consider for example a singularity of type D_n, n even. The blow-ups in the singular lines on the divisors D_i in the higher dimensional case correspond here to blow-ups in the intersections $D'_i \cap D''_i$. Performing these unnecessary extra blow-ups yields just another log resolution, and the formula for the contribution of the singularity for that log resolution will be exactly the evaluation of the formula from the first part of the proof for $m = 4$ (notice for instance that the Hodge-Deligne polynomial for D_i° becomes $2w^2 - 2w$ for $m = 4$ and the Hodge-Deligne polynomials for $(D'_i)^\circ$ and $(D''_i)^\circ$ will both be $w^2 - w$).

- For $m = 3$ it can be checked easily that the formulae are correct but again we give a more conceptual explanation. Compared with the higher dimensional case, all divisors except the last one split into two (distinct) components in the A_n case, for odd n. This is consistent with the Hodge-Deligne polynomials from (3.3), evaluated for $m = 3$. For even n, we must notice that the last blow-up is unnecessary for surfaces; performing it anyway does not yield a crepant resolution any more (the last divisor has discrepancy coefficient 1, as it should be, according to (4.2)). This last divisor is irreducible and the first $\frac{n}{2}$ blow-ups each add two components to the exceptional locus (compare this with (3.3) again).

  For the D_n case, the analogue of blowing up in a singular line on a divisor D_i would be to blow up in D_i itself, because it is just a line for $m = 3$. Such a blow-up is an isomorphism, and the result is that the divisors D_i are renamed as G_i. As intersection diagram one finds the same as in the higher dimensional case, but without the divisors D_i. To be able to compare this to (3.4), we must notice that it is logical to set $a_1 = w + 1$, $c_1 = 0$ and $b_1 = 1$ in (3.1). Then indeed all Hodge-Deligne polynomials that describe a piece of a divisor D_i are 0 in (3.4) for $m = 3$. For the E cases the same sort of arguments apply. ■

5.2. From now on, let X be a projective algebraic variety with at most (a finite number of) A-D-E singularities. Since the next results are trivial for surfaces, we will assume that $\dim X \geq 3$.

Proposition. *The stringy E-function of X is a polynomial if and only if* $\dim X = 3$ *and X has singularities of type A_n (n odd) and/or D_n (n even).*

Proof: It follows from theorem (5.1) that the contributions of the singular points for $m \geq 5$ can be written in the following form:

$$1+\frac{w^2(w^\alpha+a_{\alpha-1}w^{\alpha-1}+\cdots+a_0)}{w^{\alpha+1}+w^\alpha+\cdots+1},$$

where $\alpha \in \mathbb{Z}_{>0}$ and all $a_i \in \mathbb{Z}_{\geq 0}$. Such expressions or finite sums of such expressions can never be polynomials. For $m=4$ the contributions are given in the following table.

Type of singularity		Contribution of singular point
A_n	n even, $n=2k$	$1+\frac{w^2(w^{2k+2}-w^{k+2}+w^k-1)}{w^{2k+3}-1}$
	n odd, $n=2k-1$	$w+1$
D_n	n even, $n=2k$	$2w+1$
	n odd, $n=2k+1$	$w+1+\frac{w^2(w^{2k}-w^{k+1}+w^{k-1}-1)}{w^{2k+1}-1}$
E_6		$1+\frac{w^2(2w^6-2w^5+w^4-w^2+2w-2)}{w^7-1}$
E_7		$w+1+\frac{w^2(w^4-w^3+w-1)}{w^5-1}$
E_8		$1+\frac{w^2(2w^7-w^6-w^5+2w^4-2w^3+w^2+w-2)}{w^8-1}$

There are exactly two contributions that are polynomials and one sees again that adding a finite number of the non-polynomial expressions never gives a polynomial. ■

Theorem. *Let X be a three-dimensional projective variety with at most singularities of type A_n (n odd) and/or D_n (n even). Then the stringy Hodge numbers of X are nonnegative.*

Proof: Let us first consider the case where X has one singularity of type A_n (n odd). Denote by X_{ns} the nonsingular part of X, and let $\varphi : \widetilde{X} \to X$ be the log resolution as constructed in section 2. Then the stringy E-function of X will be $E_{st}(X) = H(X_{ns}) + uv + 1$ and the Hodge-Deligne polynomial of $\widetilde{X}$ is $H(\widetilde{X}) = H(X_{ns}) + \frac{n+1}{2}(uv)^2 + \frac{n+3}{2}(uv) + 1$. The exceptional locus counts $\frac{n+1}{2}$ components $D_1, \ldots, D_{\frac{n+1}{2}}$ whose classes in $H^2(\widetilde{X}, \mathbb{C})$ are linearly independent. This can be seen as follows. We embed $\widetilde{X}$ in a $\mathbb{P}^N$ and we intersect with a suitable hyperplane Y. Thanks to Grauert's contractibility criterion the intersection matrix of the curves $D_1 \cap Y, \ldots, D_{\frac{n+1}{2}} \cap Y$ is negative definite, and thus the classes of these curves are linearly independent in $H^2(\widetilde{X} \cap Y, \mathbb{C})$. The weak Lefschetz theorem implies then that the classes of $D_1, \ldots, D_{\frac{n+1}{2}}$ are linearly independent in $H^2(\widetilde{X}, \mathbb{C})$. Actually, these classes are all contained in $H^{1,1}(\widetilde{X})$ ([GH, p.163]). This means that $h^{1,1}(\widetilde{X}) = h^{2,2}(\widetilde{X}) \geq \frac{n+1}{2}$. Thus the coefficients of $(uv)^2$ and uv in $H(X_{ns})$ are ≥ 0 and ≥ -1 respectively. Note also that the constant term of $H(X_{ns})$ will be zero and for all other coefficients $a_{p,q}$ of $u^p v^q$ in $H(X_{ns})$, $(-1)^{p+q} a_{p,q}$ will be ≥ 0, since this is the case in $H(\widetilde{X})$. This implies that the stringy Hodge numbers of X are nonnegative.

If X has one singularity of type D_n (n even), we can choose to start from the log resolution constructed by Dais and Roczen ([DR, Section 2]) which yields $H(\widetilde{X}) = H(X_{ns}) + \frac{3n-2}{2}(uv)^2 + \frac{3n+2}{2}(uv) + 1$ with $\frac{3n-2}{2}$ components in the exceptional locus or we can use the log resolution analogous to section 2, which gives $H(\widetilde{X}) = H(X_{ns}) + (2n-2)(uv)^2 + 2n(uv) + 1$ with $2n-2$ components in the exceptional locus and then apply the same argument.

It is clear that nothing essential changes when there is more than one singularity. ■

Example. Consider the variety $X = \{xyz + t^3 + w^3 = 0\} \subset \mathbb{P}^4$, where we use coordinates (x, y, z, t, w). It is clear that the points $(1,0,0,0,0), (0,1,0,0,0)$ and $(0,0,1,0,0)$ are three-dimensional D_4 singularities. Thus, their contribution to the stringy E-function of X is $3(2w+1)$. To calculate the Hodge-Deligne polynomial of X, we divide X in three locally closed pieces:

$$X = (X \cap \{x \neq 0, y \neq 0\}) \sqcup (X \cap \{x \neq 0, y = 0\}) \sqcup (X \cap \{x = 0\}).$$

The Hodge-Deligne polynomial of the first piece is just $(w-1)w^2$ since y, z, t, w have become affine coordinates and y, t, w can be chosen freely, with $y \neq 0$. The second piece consists of three planes in $\mathbb{A}^3$, intersecting in a line and has Hodge-Deligne polynomial $3(w^2 - w) + w$ and the third piece are three planes in $\mathbb{P}^3$, in-

tersecting in a line, with contribution $3w^2+w+1$. Thus $H(X) = w^3+5w^2-w+1$ and $H(X_{ns}) = w^3 + 5w^2 - w - 2$. It follows that the stringy E-function of X is equal to $w^3 + 5w^2 + 5w + 1$ and that the stringy Hodge numbers of X are nonnegative.

Acknowledgement: I wish to thank Professor J. Steenbrink for his contribution to the proof of Theorem 5.2.

References

[Ba1] V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Proc. Taniguchi Symposium 1997, In 'Integrable Systems and Algebraic Geometry, Kobe/Kyoto 1997', World Sci. Publ. (1999), 1-32.

[Ba2] V. Batyrev, *Non-Archimedian integrals and stringy Euler numbers of log terminal pairs*, J. Europ. Math. Soc. **1** (1999), 5-33.

[Da] D. Dais, *On the string-theoretic Euler number of a class of absolutely isolated singularities*, Manuscripta Math. **105** (2001), 143-174.

[DR] D. Dais and M. Roczen, *On the string-theoretic Euler number of 3-dimensional A-D-E singularities*, Adv. Geom. **1** (2001), 373-426.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons (1978).

[Re] M. Reid, *Young person's guide to canonical singularities*, Algebraic Geometry Bowdoin 1985, Proc. Sympos. Pure Math., vol. 46 (1987), 345-414.

[Sr] V. Srinivas, *The Hodge Characteristic*, preprint.

[Ve1] W. Veys, *Arc spaces, motivic integration and stringy invariants*, to appear in Adv. Stud. Pure Math., Proceedings of "Singularity Theory and its applications, Sapporo, 16-25 september 2003".

[Ve2] W. Veys, *Stringy invariants of normal surfaces*, J. Alg. Geom. **13** (2004), 115-141.